

# STABLE MIXING FOR CAT MAPS AND QUASI-MORPHISMS OF THE MODULAR GROUP

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## 1. INTRODUCTION

It is well-known that the action of a hyperbolic element (“cat map”) of the modular group  $SL(2, \mathbf{Z})$  on the torus  $\mathbf{T} = \mathbf{R}^2/\mathbf{Z}^2$  has strong chaotic dynamical properties such as mixing and exponential decay of correlations. In this note we study stability of this behaviour with respect to a class of perturbations called “kicked systems” introduced in [PR]: Let  $h \in SL(2, \mathbf{Z})$  be hyperbolic, and assume that the system  $\{h^n\}$ ,  $n \in \mathbf{N}$  is influenced by a sequence of kicks  $\Phi = \{\phi_1, \phi_2, \dots\} \subset SL(2, \mathbf{Z})$  which arrive periodically with period  $t \in \mathbf{N}$ . The evolution of the kicked system is given by the sequence of maps

$$(1) \quad f^{(n)}(t) = \phi_n h^t \dots \phi_2 h^t \phi_1 h^t.$$

We shall always assume that the sequence  $\text{trace}(\phi_i)$  is bounded.

The kicked system defined in (1), is a particular case of the more general notion of a sequential dynamical system. Any sequence of maps  $f^{(n)}$  gives rise to dynamics on the torus, the integer  $n$  playing the role of discrete time, and the evolution of an observed quantity, that is a function  $F \in L^2(\mathbf{T})$  is given by  $F \circ f^{(n)}$ . Given two observables  $F_1, F_2 \in L^2(\mathbf{T})$ , the time correlation function is

$$(2) \quad C(F_1, F_2; f^{(n)}) = \int_{\mathbf{T}} F_1(f^{(n)}(x)) F_2(x) dx - \int_{\mathbf{T}} F_1(x) dx \int_{\mathbf{T}} F_2(x) dx.$$

The system is *mixing* if  $C(F_1, F_2; f^{(n)}) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $F_1, F_2 \in L^2(\mathbf{T})$ . For instance, the sequence  $\{h^n\}$ ,  $n \in \mathbf{N}$  is mixing if and only if  $h$  is hyperbolic.

**Definition 1.** *An element  $h \in SL(2, \mathbf{Z})$  is stably mixing if for every sequence of kicks  $\Phi = \{\phi_i\}$  with bounded traces there exists  $t_0 = t_0(\Phi)$  so*

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that for every  $t > t_0$ , the sequential system  $f^{(n)}(t) = \phi_n h^t \dots \phi_2 h^t \phi_1 h^t$  is mixing.

As it was noticed in [PR] if  $h$  is conjugate to its inverse in  $SL(2, \mathbf{Z})$ , it is *not* stably mixing, since if  $ghg^{-1} = h^{-1}$  and we take as the sequence of kicks  $g^{-1}, g, g^{-1}, g, \dots$  then the evolution of the kicked system  $f^{(n)}(t)$  is 2-periodic:  $f^{(n)}(t)$  is either  $g^{-1}h^t$  or 1 depending if  $n$  is odd or even.

In this paper we give a complete characterization for stable mixing for the 2-torus:

**Theorem 2.** *A hyperbolic element  $h \in SL(2, \mathbf{Z})$  is stably mixing if and only if it is not conjugate to its inverse in  $SL(2, \mathbf{Z})$ .*

Moreover we give a quantitative measure of mixing by showing (Theorem 5 below) that the correlations decay exponentially for Hölder observables, as well as establish mixing for a more general class of perturbations (see §4). In the last section we study “Lyapunov exponents” of kicked systems.

The proof of Theorem 2 is a combination of basic harmonic analysis and geometric group theory. The essential notion that we need is that of a *quasi-morphism* (see e.g. [BG]): Given a group  $G$ , and a function  $r : G \rightarrow \mathbf{R}$ , we say that  $r$  is a *quasi-morphism* if its *derivative*  $dr : G \times G \rightarrow \mathbf{R}$  defined by

$$dr(g_1, g_2) = r(g_1 g_2) - r(g_1) - r(g_2)$$

is a *bounded* function. A *homogeneous* quasi-morphism also satisfies  $r(g^n) = nr(g)$  for all  $n \in \mathbf{Z}$ .

In our previous work [PR, 1.5.D] (see also [EF],[BF]), we showed that given a hyperbolic element  $h$  of  $SL(2, \mathbf{Z})$ , not conjugate to its inverse, there is a homogeneous quasi-morphism of  $SL(2, \mathbf{Z})$  which does not vanish on  $h$ . Crucial to our purpose is the following refinement of that result:

**Proposition 3.** *Let  $h \in SL(2, \mathbf{Z})$  be hyperbolic, not conjugate to its inverse. Then there exists a homogeneous quasi-morphism  $r$  of  $SL(2, \mathbf{Z})$  such that  $r(h) = 1$  and which vanishes on all parabolic elements.*

This is a special case of the following “separation theorem”:

**Theorem 4.** *Let  $G$  be a discrete subgroup of  $PSL(2, \mathbf{R})$  and  $h \in G$  a primitive element of infinite order, not conjugate to its inverse in  $G$ . Then given any finite set of elements  $g_1, \dots, g_M \in G$ , not conjugate to a power of  $h$ , there is a homogeneous quasi-morphism  $r$  so that  $r(h) \neq 0$ , while  $r(g_1) = \dots = r(g_M) = 0$ .*

The existence of such a rich set of quasi-morphisms is a remarkable property of discrete groups. For instance,  $\mathbf{Z}^n$  fails to satisfy it for  $n \geq 2$  (all homogeneous quasi-morphisms are homomorphisms in that case). Moreover, for  $SL(n, \mathbf{Z})$ ,  $n \geq 3$  there are *no* nontrivial quasi-morphisms! This is a consequence of the bounded generation property - every matrix in  $SL(n, \mathbf{Z})$  is a product of a bounded number of elementary matrices<sup>1</sup> if  $n \geq 3$  [CK1, CK2] and for  $n \geq 3$  elementary matrices are commutators. See [BM1, BM2] for a generalization to lattices in higher-rank groups. It is for this reason that we work in dimension 2. It would be interesting to explore stable mixing in higher dimensions.

We will prove Theorem 4 with the use of elementary hyperbolic geometry. It seems to be likely that the separation property stated in the theorem can be extended to general hyperbolic groups – the proof could be probably extracted from a recent work by Bestvina and Fujiwara [BF].

Dima Burago and David Kazhdan pointed out to us that if we restrict the class of kicks to sequences  $\Phi$  taking a *finite* number of values, then it is possible to show that the kicked system (1) is mixing for  $t \gg 1$  under a more general condition than in Theorem 2: One need only assume that the kicks do not exchange the stable and unstable subspaces of the hyperbolic matrix  $h$ . Their argument is of a dynamical nature, and does not use quasi-morphisms. No such argument is known to us in the more general case of kicks with bounded traces as in Theorem 2. In fact, stable mixing is sensitive to the size of the kicks. For instance, stability disappears when one allows certain unbounded sequence of kicks (see Section 1.2 of [PR]). It would be interesting to determine the critical size of perturbations for which mixing persists. We refer to §4 for further discussion on this issue in terms of a special geometry on  $SL(2, \mathbf{Z})$ .

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## 2. THE SEPARATION THEOREM

**2.1. Background on quasi-morphisms.** Let  $G$  be a group. Starting with any quasi-morphism  $r'$  on  $G$ , we get a homogeneous one  $r$  by setting

$$r(g) = \lim_{n \rightarrow \infty} \frac{r'(g^n)}{n}$$

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<sup>1</sup>An elementary matrix has 1's on the diagonal, while all but one off-diagonal entries vanish.

(the limit exists because  $|r'(g^n)|$  is a subadditive sequence). A homogeneous quasi-morphism is automatically invariant under conjugation since for all  $n$

$$r(ghg^{-1}) = \frac{r((ghg^{-1})^n)}{n} = \frac{r(gh^n g^{-1})}{n} = \frac{r(g) + nr(h) - r(g) + O(1)}{n}$$

and taking  $n \rightarrow \infty$  we get  $r(ghg^{-1}) = r(h)$ .

**2.2. Quasi-morphisms for discrete subgroups of  $PSL(2, \mathbf{R})$ .** Consider the action of a discrete group  $G \subset PSL(2, \mathbf{R})$  on the upper half-plane  $\mathbf{H}$ . To construct quasi-morphisms of  $G$ , start with a smooth  $G$ -invariant one-form  $\alpha$ , such that  $|d\alpha/\Omega| \leq C$  for some  $C > 0$ , where  $\Omega = y^{-2}dx \wedge dy$  is the hyperbolic area form. Given such  $\alpha$  and a base-point  $z \in \mathbf{H}$ , set

$$r_z(g) = \int_{\ell(z, gz)} \alpha$$

where  $\ell(z, w)$  denotes the geodesic segment between two points  $z, w \in \mathbf{H}$ . As is well known,  $r_z$  is a quasi-morphism of  $G$  (the reason that  $dr_z$  is *bounded* has to do with the fact that the area of a hyperbolic triangle is bounded - see e.g. [PR, Lemma 3.2.B.]). The quasi-morphism  $r_z$  depends on  $z$  only up to a bounded quantity. Thus its homogenization

$$r(g) := \lim_n r_z(g^n)/n$$

is independent of the choice of base-point  $z$ .

**2.3. Proof of the separation theorem.** We will construct  $\alpha$  so that for our fixed element  $h$  and some point  $z_0 \in \mathbf{H}$ ,  $r_{z_0}(h^n) = n$ . Hence for the homogenization  $r$  one has  $r(h) = 1$ . Further, we make the support of  $\alpha$  sufficiently small so that for each of the elements  $g_j$ ,  $j \geq 1$ , there is a base-point  $z_j \in \mathbf{H}$  such that every geodesic segment  $\ell(z_j, g_j^n z_j)$  is disjoint from the support of  $\alpha$ . In that case  $r_{z_j}(g_j^n) = \int_{\ell(z_j, g_j^n z_j)} \alpha = 0$  and thus  $r(g_j) = 0$ . This will conclude the proof of the Theorem.

We may assume that none of the elements  $g_j$  are elliptic, since these are annihilated by any homogeneous quasi-morphism. Thus we take  $g_1, \dots, g_a$  to be hyperbolic, and  $g_{a+1}, \dots, g_M$  parabolic. Let  $L_1, \dots, L_a$  be the invariant geodesics of the hyperbolic elements  $g_1, \dots, g_a$ . Let  $p_{a+1}, \dots, p_M \in \mathbf{R} \cup \infty$  be the cusps (fixed points) of the parabolic elements  $g_{a+1}, \dots, g_M$ .

**Case 1:  $h$  primitive parabolic.** In this case  $h$  is not conjugate to its inverse already in  $PSL(2, \mathbf{R})$ . After conjugating the group  $G$ , we may assume that  $h(z) = z + 1$ . Since  $G$  is discrete,  $\infty$  is not a fixed point of any hyperbolic element of  $G$ . Therefore if  $R > 1$  is sufficiently large, the horoball  $B = \{z = x + iy : y > R\}$  is disjoint from  $\gamma B$

when  $\gamma \in G$  is not a power of  $h$ . Moreover, if  $R > 1$  is sufficiently large then  $B$  contains none of the  $G$ -translates of the geodesics  $L_j$ ,  $1 \leq j \leq a$ . Furthermore, since the parabolic elements  $g_{a+1}, \dots, g_M$  are not conjugate to a power of the primitive element  $h$ , their fixed points  $p_j$  are not  $G$ -translates of  $\infty$ . Thus (further increasing  $R$  if necessary) there exist small horoballs  $B_j \subset \mathbf{H}$  tangent to  $\mathbf{R}$  at  $p_j$  which are disjoint from all  $G$ -translates of the horoball  $B$ .

We take a smooth cutoff function  $u(y)$  which vanishes for  $y < R$  and is identically 1 for  $y > R+1$ , and set  $\alpha' = u(y)dx$ . Clearly  $|d\alpha'/\Omega| \leq C$ . Take  $\alpha$  to be the  $G$ -periodization of  $\alpha'$ , that is

$$\alpha = \sum_{\gamma \in \langle h \rangle \backslash G} \gamma^* \alpha'.$$

This is a smooth one-form, supported in the union of  $G$ -translates of the horoball  $B$ . Thus its support is disjoint from the geodesics  $L_j$  and the horoballs  $B_j$ . For  $j = 1, \dots, M$  pick a point  $z_j$  which lies on  $L_j$  for  $j \leq a$  and inside  $B_j$  for  $j > a$ . Thus  $r_{z_j}(g_j) = 0$  for all  $1 \leq j \leq M$ . To compute  $r(h)$ , take a point  $z_0$  with  $\text{Im}(z_0) = R+1$ . Then  $h^n z_0 = z_0 + n$  and  $\int_{\ell(z_0, h^n z_0)} \alpha = n$ , and so  $r_{z_0}(h^n) = n$ . This proves the claim for parabolic  $h$ .

**Case 2:  $h$  primitive hyperbolic.** Let  $L$  be the invariant geodesic of  $h$ . Since  $g_1, \dots, g_a$  are not conjugate to a power of the primitive element  $h$ , their invariant geodesics  $L_j$  are not  $G$ -equivalent to  $L$ . Let  $D$  be a locally finite fundamental domain for  $G$  (e.g. the Dirichlet fundamental domain) whose intersection with  $L$  contains a geodesic segment, say  $I$ . There are only finitely many  $G$ -translates of  $L_j$  which meet  $D$  [Be, Theorem 9.2.8(iii)], and so the intersection of  $L \cap D$  with the  $G$ -translates of the  $L_j$ 's is a finite set of points. Shrinking  $I$ , we can assume that  $I$  does not meet any of these intersection points, and moreover that a small neighborhood  $U$  of  $I$  is disjoint from the  $G$ -images of  $L_j$ . Shrinking further  $I$  and  $U$  if necessary we can achieve the following. First,  $\gamma U \cap U = \emptyset$  if  $1 \neq \gamma \in G$ . Further, since  $h$  is not conjugate to  $h^{-1}$  in  $G$ , this implies that if  $\gamma \in G$ ,  $\gamma \neq h^k$  for some  $k \in \mathbf{Z}$  then  $\gamma U$  is bounded away from  $L$  [PR, Lemma 3.2.D]. Finally, we can choose small horoballs  $B_j$  around the cusps  $p_j$  which are disjoint from  $\cup_{\gamma \in G} \gamma U$ .

Choose now a one-form  $\alpha'$  in  $\mathbf{H}$ , supported in  $U$  so that  $\int_I \alpha' = 1$ . Then  $|d\alpha'/\Omega|$  is bounded. Let  $\alpha = \sum_{\gamma \in G} \gamma^* \alpha'$  be its  $G$ -periodization. Then the support of  $\alpha$  lies in the union of  $G$ -translates of  $U$ , and so is disjoint from the geodesics  $L_j$  as well as from the horoballs  $B_j$ . For  $j = 1, \dots, M$  pick a point  $z_j$  which lies on  $L_j$  when  $j \leq a$  and inside  $B_j$  for  $j > a$ . Pick a point  $z_0$  on  $L$ . Clearly,  $r_{z_j}(g_j) = 0$  for  $j \geq 1$ . Finally

note that  $\int_{\ell(z_0, h^n z_0)} \alpha = n \int_I \alpha = n$  and so  $r_{z_0}(h^n) = n$ . This concludes the claim for the hyperbolic case.  $\square$

**2.4. Proof of Proposition 3.** Let  $\pi : SL(2, \mathbf{Z}) \rightarrow PSL(2, \mathbf{Z})$  be the projection. Assume  $h \in SL(2, \mathbf{Z})$  is hyperbolic and not conjugate to its inverse in  $SL(2, \mathbf{Z})$ . Note that  $h$  is not conjugate to  $-h^{-1}$  since  $\text{trace}(h) \neq 0$ . Let  $\tilde{r}$  be a homogeneous quasi-morphism of  $PSL(2, \mathbf{Z})$  which is 1 on  $\pi(h)$  and zero on  $\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$  whose existence is guaranteed by Theorem 4, and let  $r = \tilde{r} \circ \pi$ . Since all primitive parabolic elements are conjugate in  $SL(2, \mathbf{Z})$  to either  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  or its inverse, it follows that  $r$  vanishes on all parabolic elements of  $SL(2, \mathbf{Z})$ .  $\square$

**2.5. When a matrix is conjugate to its inverse?** Every symmetric matrix from  $SL(2, \mathbf{Z})$  is conjugate to its inverse by an element  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Further, one can prove the following statement, which is left as an exercise: For a hyperbolic matrix  $h$ , consider the prime decomposition of  $(\text{trace}(h))^2 - 4$ . If this decomposition contains a prime which equals 3 modulo 4 and enters in an odd power then  $h$  is not conjugate to  $h^{-1}$ . For instance,

$$h = \begin{pmatrix} 4 & 9 \\ 7 & 16 \end{pmatrix}$$

is not conjugate to its inverse (see also [BR]) and thus is stably mixing. There is an algorithm, based on the theory of Pell's equation, which establishes when a given matrix from  $SL(2, \mathbf{Z})$  is conjugate to its inverse, see [G, L].

### 3. MIXING AND EXPONENTIAL DECAY OF CORRELATIONS

**3.1. Decay of correlations.** In this section we prove Theorem 2 as well as a quantitative version giving an exponential rate of decay of the correlations functions  $C(F_1, F_2; f^{(n)})$  given by (2). Mixing means that the correlation functions decay as  $n \rightarrow \infty$  for  $L^2$  observables. To get a rate of decay, we have to restrict the observables. A function  $F$  on the torus satisfies *Hölder's condition* if for some  $\gamma > 0$  there is a positive constant  $c > 0$  so that for all  $x, y \in \mathbf{T}$  we have

$$(3) \quad |F(x) - F(y)| \leq c \|x - y\|^\gamma.$$

We will show that our kicked systems have exponential decay of correlations for Hölder observables.

**Theorem 5.** *If  $h \in SL(2, \mathbf{Z})$  is a hyperbolic and not conjugate to its inverse, there is some  $t_0$  so that for all  $t > t_0$  and for all Hölder functions  $F_1, F_2$  on  $\mathbf{T}$  the correlations  $C(F_1, F_2; f^{(n)}(t))$  decay exponentially:*

$$|C(F_1, F_2; f^{(n)}(t))| \leq ce^{-\gamma n}$$

for some  $c, \gamma > 0$ .

In order to prove theorem 5 we start with a general observation:

**Proposition 6.** *Let  $f^{(n)} \in SL(2, \mathbf{Z})$  be a sequential system. If there are  $\alpha > 0$ ,  $\beta > 1$  such that for every integer vector  $0 \neq v \in \mathbf{Z}^2$  and every  $n \geq 1$  we have*

$$\|v f^{(n)}\| \geq \alpha \frac{\beta^n}{\|v\|}$$

then the system is mixing and moreover we have exponential decay of correlations for all Hölder observables.

*Proof.* It clearly suffices to deal with the case where  $F_1 = F_2$  and both have mean zero. Let

$$(4) \quad N \leq \sqrt{\alpha \beta^n}.$$

Take a function in  $L^2(\mathbf{T})$ , of mean zero, and consider its Fourier expansion

$$F(x) = \sum_{0 \neq v \in \mathbf{Z}^2} a(v) e^{2\pi i(v, x)} = F_N(x) + R_N(x)$$

where  $F_N(x) = \sum_{\|v\| < N} a(v) e^{2\pi i(v, x)}$  is the partial sum over all frequencies of norm less than  $N$ . Then the correlation function  $C(F, F; f^{(n)})$  is a sum of terms

$$\begin{aligned} C(F, F; f^{(n)}) &= \int_{\mathbf{T}} F_N(f^{(n)}x) F_N(x) dx \\ &\quad + \int_{\mathbf{T}} F_N(f^{(n)}x) R_N(x) dx + \int_{\mathbf{T}} R_N(f^{(n)}x) F(x) dx. \end{aligned}$$

By Cauchy-Schwartz, the second and third terms above are bounded respectively by

$$\|F_N \circ f^{(n)}\|_2 \|R_N\|_2 = \|F_N\|_2 \|R_N\|_2 \leq \|F\|_2 \|R_N\|_2$$

and by

$$\|R_N \circ f^{(n)}\|_2 \|F\|_2 = \|R_N\|_2 \|F\|_2,$$

where  $\|F\|_2$  stands for the  $L^2$ -norm. The first term equals

$$\int_{\mathbf{T}} F_N \circ f^{(n)}(x) F_N(x) dx = \sum_{\substack{\|v f^{(n)}\| < N \\ \|v\| < N}} a(-v f^{(n)}) a(v).$$

We claim that this is an empty sum, hence vanishes, for our choice of  $N$ : Indeed, by our condition if  $0 < \|v\| < N$  we have

$$\|-vf^{(n)}\| \geq \alpha \frac{\beta^n}{\|v\|} > \frac{\alpha\beta^n}{N} \geq N$$

since  $N \leq \sqrt{\alpha\beta^n}$ . Thus we find that for  $N$  as above that

$$|C(F, F; f^{(n)})| \leq 2\|F\|_2\|R_N\|_2.$$

Since we can choose  $N \rightarrow \infty$  as  $n \rightarrow \infty$  (subject to (4)) we get  $\|R_N\|_2 \rightarrow 0$  and hence we have mixing.

To prove *exponential* decay of correlations for Hölder observables  $F$ , recall that if  $F$  satisfies the Hölder condition (3) of order  $\gamma > 0$  then for some constant  $c_F > 0$

$$\|R_N\|_2 \leq c_F N^{-\gamma}$$

(see e.g. [Z, Vol. I, Chapter II.4] for the proof in the case of functions of one variable) and hence taking  $N$  to be the smallest integer less than  $\sqrt{\alpha\beta^n}$  gives  $\|R_N\|_2 \leq \text{const } \beta^{-n\gamma/2}$  as required.  $\square$

**3.2. Using quasi-morphisms.** For a quasi-morphism  $r$  put

$$\|dr\| := \sup_{g_1, g_2 \in G} |r(g_1 g_2) - r(g_1) - r(g_2)|.$$

To show that our kicked systems satisfy the condition of Proposition 6, we use

**Proposition 7.** *Let  $r$  be a homogeneous quasi-morphism of  $SL(2, \mathbf{Z})$  which vanishes on all parabolic elements. Then there is  $\alpha > 0$  such that for every integer vector  $0 \neq v \in \mathbf{Z}^2$  and every  $f \in SL(2, \mathbf{Z})$ , we have*

$$\|vA\| \geq \alpha \frac{e^{|r(A)|/\|dr\|}}{\|v\|}$$

**Proof:** A vector  $v \in \mathbf{Z}^2$  is *primitive* if it is not a nontrivial multiple of another integer vector. It clearly suffices to prove the result for primitive vectors.

**Lemma 8.** *Every primitive vector  $v \in \mathbf{Z}^2$  can be written as*

$$v = (0, 1) \begin{pmatrix} 1 & k_N \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ k_{N-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & k_2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ k_1 & 1 \end{pmatrix}$$

with  $N \leq \log_2 \|v\| + 10$ .



*Proof.* This is the well known Euclidean algorithm: Assume that  $v = (p, q)$  with  $|q| \geq |p|$ . Primitivity is equivalent to  $p, q$  being co-prime. Then find an integer  $k_1$  so that  $q = k_1 p + r_1$  with  $|r_1| \leq |p|/2$ . We then arrive at a new vector  $v_2 := (p, r_1) = (p, q) \begin{pmatrix} 1 & -k_1 \\ 0 & 1 \end{pmatrix}$ . Now find an integer  $k_2$  so that  $p = k_2 r_1 + r_2$  with  $|r_2| \leq |r_1|/2$  to get another vector  $v_3 := (r_2, r_1) = (p, r_1) \begin{pmatrix} 1 & 0 \\ -k_2 & 1 \end{pmatrix}$  and so on. This proceeds until we get to the point that we have computed the greatest common divisor of  $p$  and  $q$  (which in our case equals 1) with at most  $\log_2 |p| + 1 \leq \log_2 \|v\| + 1$  steps. This gives us either the vector  $(0, 1)$ , in which case we are done, or the vector  $(1, 0)$ . In the latter case just note that  $(0, 1) = (1, 0) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ .  $\square$

By lemma 8 every primitive vector is of the form  $w = (0, 1)h$  with  $h$  a product of at most  $\log_2 \|w\| + 10$  elementary matrices. The quasi-morphism  $r$  vanishes on these matrices and so we have  $|r(h)| \leq (\log_2 \|w\| + 10)\|dr\|$ . Apply this reasoning to the vectors  $v = (0, 1)h_1$  and  $vf^{(n)} = (0, 1)h_2$ . Then  $|r(h_1)| \leq (\log_2 \|v\| + 10)\|dr\|$  and  $|r(h_2)| \leq (\log_2 \|vf^{(n)}\| + 10)\|dr\|$ . The matrix  $h_3 := h_1 f h_2^{-1}$  is parabolic, since it fixes  $(0, 1)$  and so

$$\begin{aligned} |r(f)| &= |r(h_1^{-1} h_2 h_3)| \leq |r(h_1)| + |r(h_2)| + 2\|dr\| \\ &\leq (\log_2 \|v\| + \log_2 \|vf\| + 22)\|dr\| \end{aligned}$$

Therefore

$$\|vf\| \geq 2^{-22} \frac{2^{|r(f)|/\|dr\|}}{\|v\|}$$

as required.  $\square$

**3.3. Proof of Theorem 5.** By Proposition 3 if  $h$  is not conjugate to its inverse then there is a homogeneous quasi-morphism  $r$  of  $SL(2, \mathbf{Z})$  which vanishes on parabolic elements and for which  $r(h) = 1$ . We claim that for any sequence of kicks  $\phi_i \in G$  with bounded traces one has  $|r(\phi_1)| \leq c$  for some  $c > 0$ . Indeed,  $r$  vanishes on elliptic and parabolic elements, while hyperbolic elements whose traces are bounded represent a *finite* number of conjugacy classes in  $G$ . The claim follows.

Further,

$$r(f^{(n)}(t)) = r\left(\prod_{i=1}^n h^t \phi_i\right) = \sum_{i=1}^n r(h^t) + r(\phi_i) + O_r(1) = nt + O_{r,\Phi}(n)$$

and so if  $t$  is big enough we have  $|r(f^{(n)}(t))| \geq n$ . By proposition 7 it follows that for all nonzero integer vectors  $v$  we have

$$\|vf^{(n)}(t)\| \geq \alpha \frac{\beta^n}{\|v\|}$$

for some  $\beta > 1$ ,  $\alpha > 0$  and thus Proposition 6 concludes the proof.  $\square$

#### 4. STABLE MIXING FROM THE GEOMETRIC VIEWPOINT

In this section we present a generalization of the stable mixing phenomenon described above. Define a biinvariant metric  $\rho$  on the group  $G = SL(2, \mathbf{Z})$  as follows. Every element  $g \in G$  can be decomposed as  $g = a_1 \dots a_d$ , where every  $a_i$  is either elliptic, or parabolic, or a simple commutator of the form  $aba^{-1}b^{-1}$ ,  $a, b \in G$ . Put  $\rho(1, g) = \inf d$ , where the infimum is taken over all such presentations, and set  $\rho(f, g) = \rho(1, fg^{-1})$  for all  $f, g \in G$ .

Denote by  $G^\infty$  the set of all infinite sequences  $\{g^{(k)}\}$ ,  $k \in \mathbf{N}$  of "moderate growth", that is of those which satisfy  $\sup_{k \in \mathbf{N}} \rho(1, g^{(k)})/k < \infty$ . For instance, for every  $h \in G$  the cyclic semigroup  $\{h^k\}$ ,  $k \in \mathbf{N}$  lies in  $G^\infty$ . Define a metric on  $G^\infty$  by

$$\bar{\rho}(\{f^{(k)}\}, \{g^{(k)}\}) = \sup_{k \in \mathbf{N}} \rho(f^{(k)}, g^{(k)}).$$

**Theorem 9.** *Let  $h \in G$  be a hyperbolic element which is not conjugate to its inverse in  $G$ . Then there exists  $\epsilon > 0$  such that the ball of radius  $\epsilon$  centered at  $\{h^k\} \in G^\infty$  consists of mixing sequences with exponential decay of correlations on Hölder observables.*

*Proof.* One can easily check that homogeneous quasi-morphisms on  $G$  which vanish on parabolics are Lipschitz functions with respect to metric  $\rho$ . Let  $r$  be such a quasi-morphism with  $r(h) = 1$ . Take  $\epsilon > 0$  sufficiently small and consider any sequence  $f = \{f^{(k)}\} \in G^\infty$  with  $\bar{\rho}(\{f^{(k)}\}, \{h^k\}) < \epsilon$ . It follows that  $|r(f^{(k)})| \geq ck$  for some  $c > 0$  and all  $k \in \mathbf{N}$ . Applying Propositions 7 and 6, we get that  $f$  is mixing with exponential decay of correlations on Hölder observables.  $\square$

**Remark.** Throughout the paper we worked with hyperbolic elements  $h$  which are not conjugate to  $h^{-1}$  in  $G$ . This assumption can be translated in the geometric language as follows. We say that a cyclic subgroup  $\{h^k\}$ ,  $k \in \mathbf{Z}$  has *linear growth* if  $\rho(1, h^k) \geq c|k|$  for some  $c > 0$ . It follows from Proposition 3 that the linear growth is equivalent to the fact that  $h$  is a hyperbolic element which is not conjugate to its inverse in  $G$ . Otherwise cyclic subgroup generated by  $h$  remains a bounded distance from the identity.

To include kicked systems into the geometric framework introduced above, we start with the following observation. Denote by  $\epsilon(h)$  the maximal value of  $\epsilon$  supplied by Theorem 9. Then for  $t \in \mathbf{N}$  one has  $\epsilon(h^t) \geq t\epsilon(h)$ . Assume now that  $h$  is not conjugate to its inverse in  $G$ , take a sequence of kicks  $\Phi = \{\phi_i\}$  with bounded traces, and consider the kicked system  $f^{(n)}(t) = \phi_n h^t \dots \phi_1 h^t$ . Note that

$$h^{nt} = f^{(n)}(t) \circ \prod_{j=1}^n h^{-tj} \phi_j^{-1} h^{tj}.$$

Therefore there exists  $c > 0$  which depends only on  $\Phi$  such that for every  $t \in \mathbf{N}$

$$\bar{\rho}(\{h^{nt}\}, \{f^{(n)}(t)\}) \leq c.$$

Hence the kicked system  $f^{(n)}(t)$  is mixing provided  $c < t\epsilon(h) \leq \epsilon(h^t)$ .

## 5. POSITIVE LYAPUNOV EXPONENT

We say that a sequential system  $\{f^{(n)}\}$  has *positive Lyapunov exponent* if the sequence  $\text{trace}(f^{(n)})$  grows exponentially with  $n$ . It turns out that kicked system (1) has positive Lyapunov exponent provided  $h$  is not conjugate to its inverse and  $t$  is large enough. Indeed, choose a homogeneous quasi-morphism  $r$  which satisfies  $r(h) = 1$  and vanishes on parabolics, and apply the following estimate.

**Theorem 10.**

$$|\text{trace}(f)| \geq (2\sqrt{5})^{|r(f)|/||dr||},$$

for every hyperbolic  $f \in SL(2, \mathbf{Z})$  and any homogeneous quasi-morphism  $r$  which vanishes on parabolics.

*Proof.* The proof is divided into 3 steps.

1) We claim that every hyperbolic matrix  $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is conjugate in  $SL(2, \mathbf{Z})$  to a matrix  $g = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$  with  $|c_1| \leq |\text{trace}(f)|/\sqrt{5}$ . Indeed, assume that  $g = hfh^{-1}$ , where  $h = \begin{pmatrix} z & t \\ x & y \end{pmatrix} \in SL(2, \mathbf{Z})$ . One calculates that  $c_1 = Q(x, y)$ , where  $Q(x, y) = cx^2 + (a - d)xy - by^2$ . The discriminant  $\Delta$  of the quadratic form  $Q$  equals  $\text{trace}^2(f) - 4$ . Since  $f$  is hyperbolic,  $\sqrt{\Delta}$  is irrational. In this case (see [Ca]) there exists a primitive vector  $(x, y) \in \mathbf{Z}^2$  with  $|Q(x, y)| \leq \sqrt{\Delta}/\sqrt{5} \leq |\text{trace}(f)|/\sqrt{5}$ . Hence we obtain  $h \in SL(2, \mathbf{Z})$  as required.

2) As a consequence we get that every hyperbolic matrix  $f \in SL(2, \mathbf{Z})$  decomposes as  $f = f'h$ , where  $|\text{trace}(f')| \leq |\text{trace}(f)|/(2\sqrt{5})$  and

$h$  is parabolic. Indeed, in view of step 1 we can assume without loss of generality that  $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $|c| \leq |\text{trace}(f)|/\sqrt{5}$ . Write  $f' = f \begin{pmatrix} 1 & k \\ & 1 \end{pmatrix}$ , so  $\text{trace}(f') = \text{trace}(f) + ck$ . One can always choose  $k \in \mathbf{Z}$  so that  $|\text{trace}(f')| \leq |c|/2 \leq |\text{trace}(f)|/(2\sqrt{5})$ .

3) For an integer  $s \geq 3$  put  $u(s) = \max |r(g)|$ , where the maximum is taken over all hyperbolic matrices  $g$  whose trace lies in  $[-s; s]$ . Since a quasi-morphism is a class function on the group, and the number of hyperbolic conjugacy classes with given trace is finite, the function  $u$  is well defined. Step 2 yields inequality  $u(s) \leq u([s/(2\sqrt{5})]) + \|dr\|$  for all  $s \geq 3$ , where we set  $u(s) = 0$  for  $s \leq 2$ , and brackets stand for the integral part of a real number. Arguing by induction we get that  $u(s) \leq \|dr\| \log_{2\sqrt{5}} s$  for all  $s \geq 3$ . This completes the proof.  $\square$

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